Capacitated Network Bargaining Games: Stability and Structure

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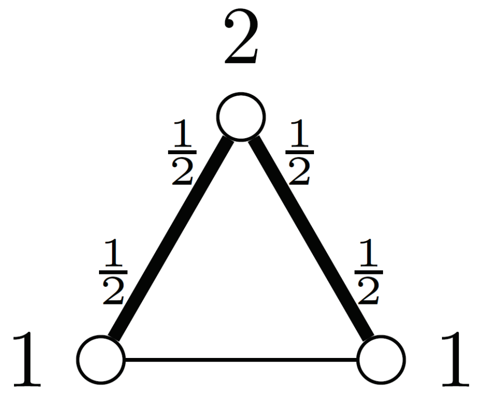
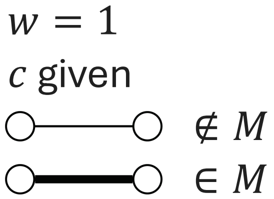
Based on joint work with Laura Sanità

Capacitated Network Bargaining Games

- Given: graph $G = (V, E)$, edge weights $w \in \mathbb{R}^E_{\geq 0}$, vertex capacities $c \in \mathbb{Z}^V_{\geq 0}$.
- V: players
- E: potential deals with value w
- A player v can enter in c_v deals: a set of deals is a c-matching M.
- Players decide how to split the value of their deal:
$$a_{uv} + a_{vu} = w_{uv} \text{ if } uv \in M \text{ and } a_{uv} = a_{vu} = 0 \text{ otherwise.}$$
- An outcome (M, a) is *stable* if no pair of players has an incentive to both leave one of their own deals to enter in a deal with each other.



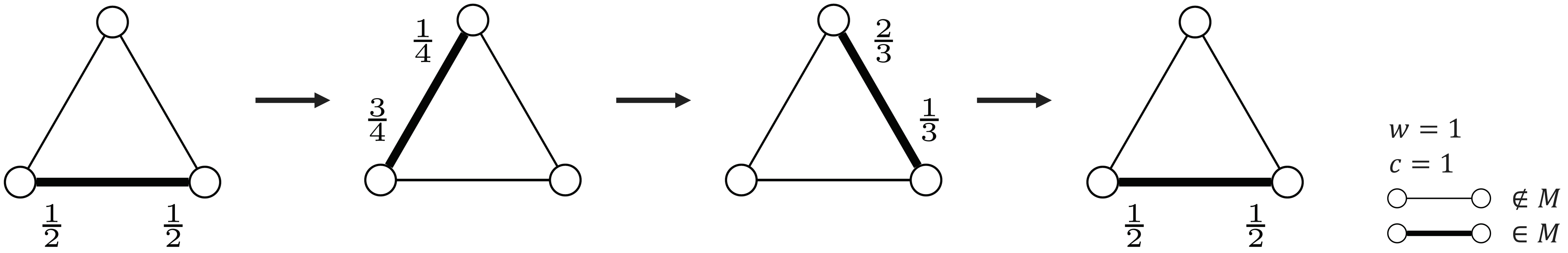


Stability

- Not all graphs allow for a stable outcome.

$\longrightarrow$ Modify the graph so that a stable outcome does exist, we call this *stabilizing*.

- We are interested in stabilizing through a minimal number of modifications.
- We consider two modifications:
  - capacity reduction,
  - edge removal.
- In unit-capacity ($c = 1$) graphs these problems are well studied:
  - Capacity reduction (or vertex removal): polynomial-time solvable [AHS18, IK+17, KS20].
  - Edge removal: hard-to-approximate with a constant factor, but there is an $O(\text{max degree})$-approximation algorithm [BC+15, KS20].



Structure

- [BH+10] There exists a stable outcome for the game defined on G if and only if there exists an integral optimal solution to the following LP
$$max\{w^T x : x(\delta(v)) \leq c_v \; \forall v \in V, 0 \leq x \leq 1\}.$$

- We call feasible solutions of this LP *fractional c-matchings*.

- Extreme point solutions of this LP satisfy $x_e \in \{0, \frac{1}{2}, 1\}$ for all edges $e \in E$, and the edges with $x_e = \frac{1}{2}$ induce vertex-disjoint odd cycles with saturated vertices.

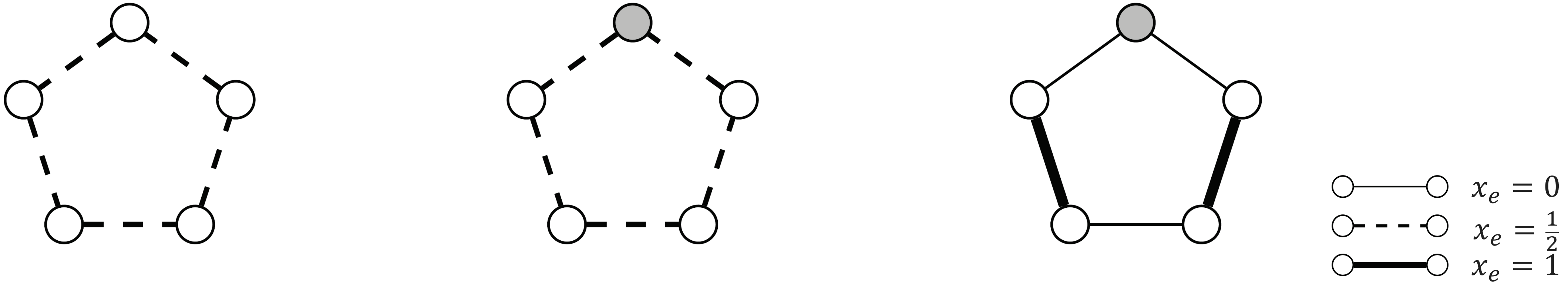
⟶ **We want to get rid of these odd cycles.**

Capacity Reduction

Theorem. Stabilizing by reducing a minimal amount of capacity is polynomial-time solvable.

- Idea of the algorithm:
  - Compute a basic maximum-weight fractional c-matching with minimum number of odd cycles induced by the $x_e = \frac{1}{2}$ edges.
  - For each such odd cycle, choose one vertex and reduce its capacity by one.

- This algorithm reduces the capacity by a minimal amount: the minimum number of these odd cycles is a lower bound on the amount of capacity that has to be reduced to stabilize. ★



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The largest possible total value of all deals that are made in an outcome (not necessarily stable).

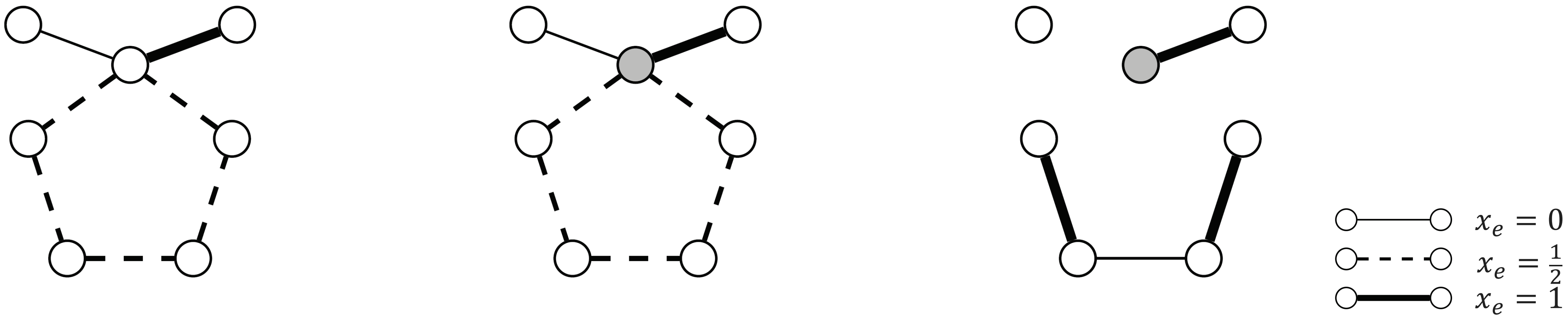
- Key properties of our algorithm:
  - It preserves at least $\frac{2}{3}$ of the total value that the players can get, which is best possible.
  - It reduces the capacity of each player by at most one.

The odd cycles are vertex disjoint.

Edge Removal

Theorem. There is an $O(\text{max degree})$-approximation algorithm for stabilizing by removing a minimal number of edges.

- Idea of the algorithm:
  - Use the capacity reduction algorithm.
  - For each vertex that would have its capacity reduced, remove all edges incident with it, except the ones with $x_e = 1$.



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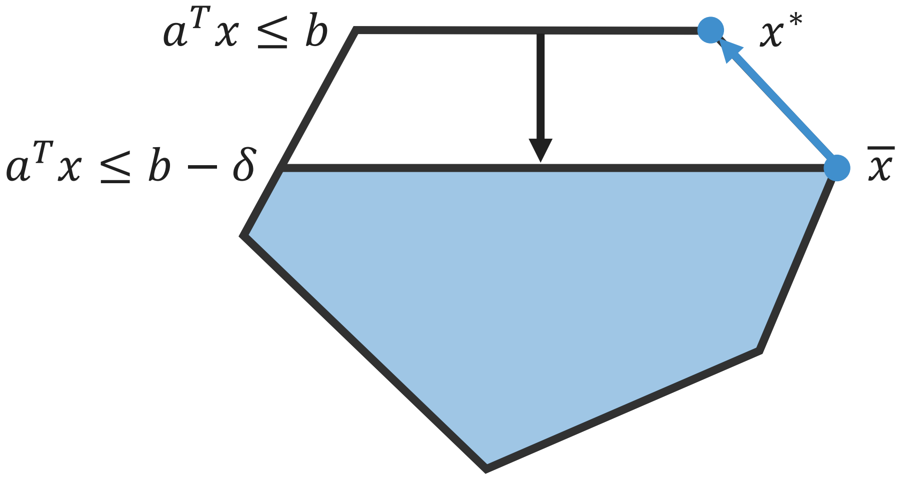
- Idea of the algorithm:
  - Use the capacity reduction algorithm.
  - For each vertex that would have its capacity reduced, remove all edges incident with it, except the ones with $x_e = 1$.

- Key property of our algorithm:
  - In unit-weight ($w = 1$) graphs, it completely preserves the total value that the players can get.

Polyhedral Tools

Theorem. Let $\mathcal{P}$ be any polytope, $a^T x \leq b$ be an inequality of te description of $\mathcal{P}$, and $\delta \in \mathbb{R}_{\geq 0}$. Let $\overline{x}$ be an optimal solution of the LP $\max\{c^T x : x \in \mathcal{P}, a^T x \leq b - \delta\}$, and assume that $\overline{x}$ is a non-optimal vertex of the LP $\max\{c^T x : x \in \mathcal{P}\}$. Furthermore, assume that there is no vertex $\tilde{x}$ of $\mathcal{P}$ satisfying $b - \delta < a^T \tilde{x} < b$. Then it is possible to move to an optimal solution x^* of $\max\{c^T x : x \in \mathcal{P}\}$ from $\overline{x}$ in one step over the edges of $\mathcal{P}$ (i.e., there is an optimal vertex of $\mathcal{P}$ adjacent to $\overline{x}$).



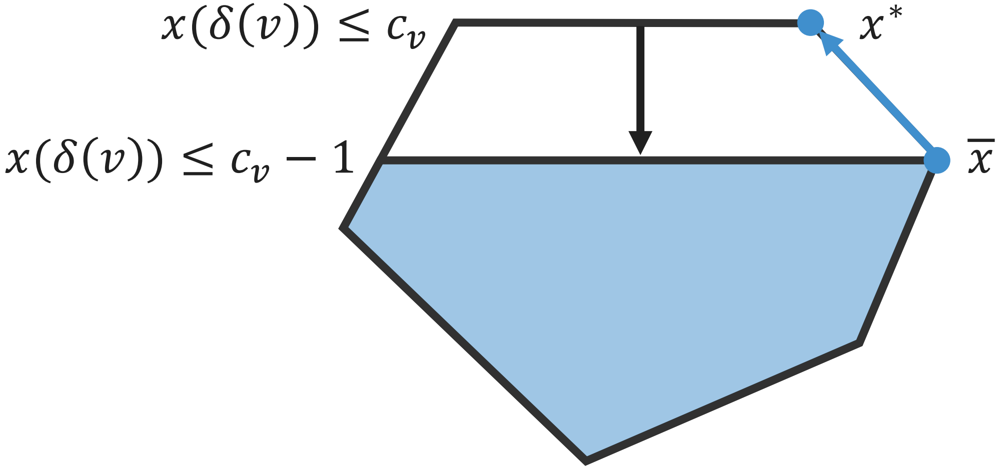
Polyhedral Tools

- Let's apply the theorem to the fractional c-matching polytope:

$$\mathcal{P} = \left\{x \in \mathbb{R}^E : x\big(\delta(v)\big) \leq c_v \; \forall v \in V, 0 \leq x \leq 1\right\}.$$

- In this polytope there is only a few ways this step can look, and all of them increase the number of odd cycles induced by the $x_e = \frac{1}{2}$ edges, by at most one.

$\Longrightarrow$ The minimum number of odd cycles induced by the $x_e = \frac{1}{2}$ edges is a lower bound on the amount of capacity that has to be reduced to stabilize.



Conclusion

- Stabilizing by reducing a minimal amount of capacity is polynomial-time solvable.
- There is an $O(\text{max degree})$-approximation algorithm for stabilizing by removing a minimal number of edges.
- We used (new) polyhedral techniques to prove our results.

Open problems

- Stabilizing by removing vertices when $c \leq 2$.
- Approximation algorithm for stabilizing by removing a minimal number of edges while simultaneously completely preserving the total value that players can get.

Thank you!

Questions?

References

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[IK+17] Ito, Kakimura, Kamiyama, Kobayashi, Okamoto. Efficient stabilization of cooperative matching games. (2017)

[KS20] Koh, Sanità. Stabilizing weighted graphs. (2020)